Singular-Value Based Stability and Sensitivity Analysis of Discrete Multiloop Systems

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Singular values and their gradients have been used to analyze the stability and sensitivity of continuous multiloop systems. This method is extended in this paper to analyze discrete systems directly in discrete domain. The discrete domain relationship between the stability margins and the minimum singular value of the return difference matrix is derived. A formulation to compute the singular value gradients is presented. This method is applied to the lateral attitude control loop of a remotely piloted vehicle both in continuous case and discrete case for verification.

Nomenclature

A,B,C,D	= continuous controller matrices
$\bar{A},\bar{B},\bar{C}$	= continous augmented system matrices
Ac,Bc,Cc,Dc	= discrete controller matrices
F,Gu,H	= plant matrices
G	= plant transfer function matrix
L	= perturbation transfer function matrix
R	= system reference input vector
T	= sampling time interval
U	= plant input vector
V	= right singular vector
X	= augmented system state vector
Xc	= controller state vector
Xs	= plant state vector
Z	= plant output vector
Γs	= integral of the transition matrix multiplied by Gu
$\bar{\sigma}$	= maximum singular value
$\underline{\sigma}$	= minimum singular value
Φs	= transition matrix
Φ,Γ,ξ	= discrete augmented system matrices

Introduction

A GOOD feedback control system should be stable and robust to modeling errors and system perturbation. In a single-input single-output (SISO) system, the robustness is built in through the classical gain margins (GM) and phase margins (PM) using Bode and Nyquist diagrams. In the case of a multi-input multioutput (MIMO) system, these methods are not adequate, because gain and phase changes can simultaneously occur in all of the loops. Singular values (SV) have recently been used to measure the stability of the MIMO system.[1-8] Singular value gradients (SVG) with respect to the variation in the system model also have been used to get sensitivity information.[1,6,8] Using the minimum SV of the system return difference matrix (RDM), Lehtomaki and Sandell[2] developed criteria for predicting stability margins of multiloop systems which are guaranteed in the case of either gain or phase simultaneous changes in all loops.[2] To get the guaranteed gain and phase margins, which include simutaneous gain and phase changes in all loops, a universal gain-phase diagram was constructed by Mukhopadhyay and Newsom.[5,6] Vaillard and Downing presented a sensitivity analysis method using SVG of RDM to judge the impact of plant dynamic model variations on the relative stability of continuous multiloop systems.[1,8] In Ref. 1, the concept of stability margins was explored using the transformation from the z plane to the w' plane for predicting the gain and phase margins of a sampled data system. In Ref. 9, SVG equations with respect to the digital controller parameter were derived and used to design a robust digital controller. Approximated SVG equations with respect to the plant parameter were derived in Ref. 10. For predicting stability margins of discrete multiloop systems, the relationship between the minimum SV of a discrete system and its stability margins was derived in this paper using the method similar to the continuous case of Ref. 2. A formulation to calculate SVG values with respect to all of the system parameters is presented using matrix gradient algebra. The results are similar to those of the continuous case.

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Review of the Singular Value Properties in the Continuous Multi-Input Multioutput System

A multiloop feedback control system is described by Eqs. (1-4).

Plant:

$$\dot{X}s = FXs + GuU \tag{1}$$

$$Z = HXs \tag{2}$$

Controller:

$$\dot{X}c = AXc + BZ \tag{3}$$

$$U = CXc + DZ + R \tag{4}$$

Equations (1-4) are written in an augmented form as

$$\dot{X} = \bar{A}X + \bar{B}U \tag{5}$$

$$U = -\bar{C}X + R \tag{6}$$

where

$$X = \begin{bmatrix} Xs \\ Xc \end{bmatrix},\ \bar{A} = \begin{bmatrix} F & 0 \\ BH & A \end{bmatrix},\ \bar{B} = \begin{bmatrix} Gu \\ 0 \end{bmatrix},\ \bar{C} = [-DH : -C]$$

Equation (6) can be written as

$$U = [I + \bar{C}(Is - \bar{A})^{-1}\bar{B}]^{-1}R \tag{7}$$

where s is the Laplace operator, and the matrix $[I + \bar{C}(Is - \bar{A})^{-1}\bar{B}]$ is called the RDM for the loop broken at the plant input, i.e.,

$$\text{RDM} = I + \bar{C}(Is - \bar{A})^{-1}\bar{B} \tag{8}$$

$$= I + \bar{C}\phi\bar{B} \tag{9}$$

where $\phi = (Is - \bar{A})^{-1}$.

It can be seen from Eq. (7) that the input vector U approaches infinity, and the system approaches instability if the RDM is close to a singularity. Also, if the RDM is near singular, the system will be sensitive to model errors, and a model error could make the system unstable. Therefore, the near singularity of the RDM may be a good measure of relative stability of the system.[1] The relationship between minimum SV and the multiloop stability margins is given by[1,2]

$$GM = [1/(1 \pm \alpha)] \tag{10}$$

$$PM = \pm \cos^{-1}[1 - (\alpha^2/2)] \tag{11}$$

where α is the minimum SV of the system RDM.

The singular value decomposition algorithm yields an SVG equation with respect to a sensitivity parameter p (Refs. 1, 6, and 8)

$$\frac{\partial \sigma_i(I + \bar{C}\phi\bar{B})}{\partial p} = \text{Re}\cdot\text{tr}\left[\left(\frac{\partial \bar{C}}{\partial p}\phi\bar{B} + \bar{C}\phi\frac{\partial \bar{B}}{\partial p} + \bar{C}\phi\frac{\partial \bar{A}}{\partial p}\phi\bar{B}\right)V_i U_i^*\right] \tag{12}$$

where $\sigma_i(\#)$ is SV of #, Re · # real of #, tr[#] trace of #, V_i and U_i are the corresponding right and left unitary singular vectors, and the superscript * means conjugate transpose. From Eq. (12) the partial derivatives with respect to each element in system matrices are

$$\frac{\partial \sigma_i}{\partial p_A} = \text{Re}\cdot\text{tr}\left[\bar{C}\phi\frac{\partial \bar{A}}{\partial p_A}\phi\bar{B}V_i U_i^*\right] \tag{13a}$$

$$\frac{\partial \sigma_i}{\partial p_B} = \text{Re}\cdot\text{tr}\left[\bar{C}\phi\frac{\partial \bar{B}}{\partial p_B}V_i U_i^*\right] \tag{13b}$$

$$\frac{\partial \sigma_i}{\partial p_C} = \text{Re}\cdot\text{tr}\left[\frac{\partial \bar{C}}{\partial p_C}\phi\bar{B}V_i U_i^*\right] \tag{13c}$$

where σ_i is $\sigma_i(I + \bar{C}\phi\bar{B})$. Using the matrix derivation equation[11,12]

$$\frac{\partial}{\partial A}\text{tr}[BAC] = [CB]^t \tag{14}$$

gives SVG with respect to all elements

$$\frac{\partial \sigma_i}{\partial A} = \text{Re}\cdot[\phi\bar{B}V_i U_i^*\bar{C}\phi]^t \tag{15a}$$

$$\frac{\partial \sigma_i}{\partial B} = \text{Re}\cdot[V_i U_i^*\bar{C}\phi]^t \tag{15b}$$

$$\frac{\partial \sigma_i}{\partial C} = \text{Re}\cdot[\phi\bar{B}V_i U_i^*]^t \tag{15c}$$

Using Eqs. (15), we can evaluate sensitivities of all elements in matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ affecting relative stabilities of continuous multiloop systems.

Extension of Singular Value Utilities to Discrete Multi-Input Multioutput Systems

For analyzing the stabilities and sensitivities of discrete MIMO systems, a relationship between the minimum singular value and stability margins is derived. An SVG calculation method is also derived. The process is very similar to that of the continuous case.[1,6]

Relationship Between Minimum Singular Value and Stability Margins in Discrete Multi-Input Multioutput Systems

The RDM of the perturbed system of Fig. 1 is represented as

$$\text{RDM} = [I + H(z)GL(z)] \tag{16}$$

where z is the z-transform operator. To decompose $GL(z)$, we need an assumption that there exists the equivalent system of Fig. 2. In Fig. 2 the perturbation $L(s)$ is represented as

$$L(s) = L'(s)^*\frac{1 - e^{-sT}}{s} \tag{17}$$

where $L'(s)^*$ is the sampling of $L'(s)$ and is given by

$$L'(s)^* = \frac{1}{T}\sum_{n=-\infty}^{\infty} L'(s + jn\omega_s)$$

and $\omega_s = 2\pi/T$. Under the preceding assumption, the RDM of the equivalent system is represented as

$$\text{RDM} = [I + H(z)G(z)L'(z)] \tag{18}$$

where $G(z) = Z[G(s)^*(1 - e^{-sT})/s]$, $L'(z) = Z[L'(s)^*(1 - e^{-sT})/s]$, and $Z[\#]$ is the Z transform of #. And $L'(Z)$ can be taken to be a diagonal matrix of Eq. (18) as in the continuous case.

$$L'(z) = \text{Diag}[l_1'(z), l_2'(z), \ldots, l_n'(z)] \tag{19}$$

$$l_i'(z) = K_i'(z)e^{j\phi'(z)} \tag{20}$$

where $K_i'(z)$ and $\phi_i'(z)$ are real and represent magnitude and phase perturbation, respectively.

If L^{-1} exists, the stability condition of the perturbed system is given by (see Ref. 2 for the continuous case)

$$\bar{\sigma}[L'(z)^{-1} - I] < \underline{\sigma}[I + H(z)G(z)] \tag{21}$$

When the minimum SV of the nominal system RDM is given by

$$\underline{\sigma}[I + H(z)G(z)] \geq \alpha_d \tag{22}$$

Equation (21) is written as

$$|l_i'^{-1}(z) - 1| < \alpha_d \tag{23}$$

For calculating gain margin, if we put

$$l_i'(z) = K_i'(z) \tag{24}$$

Eq. (23) is rearranged as

$$\frac{1}{1 + \alpha_d} < l_i'(z) < \frac{1}{1 - \alpha_d} \tag{25}$$

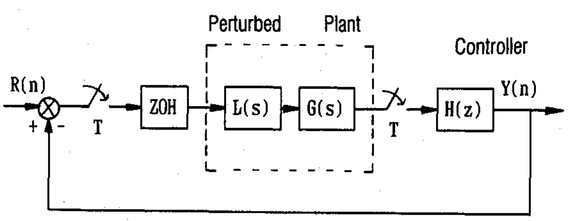

Fig. 1 Perturbed discrete MIMO system.

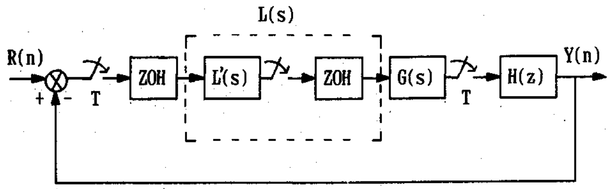

Fig. 2 Equivalent system of Fig. 1.

where $\alpha_d \leqq 1$. Using the relation

$$l'(z) = Z\left[l_i'(s) * \frac{1-e^{-sT}}{s} \right] \tag{26}$$

Eq. (25) is inverse Z transformed as

$$\frac{1}{1+\alpha_d} < l_i'(s) * \frac{1-e^{-sT}}{s} < \frac{1}{1-\alpha_d} \tag{27}$$

From Eqs. (17) and (27), we can get the perturbation interval of $l_i(s)$,

$$\frac{1}{1+\alpha_d} < l_i(s) < \frac{1}{1-\alpha_d} \tag{28}$$

and from Eq. (24), we can conceive that

$$l_i(s) = k_i(s) \tag{29}$$

because the real in the Z plane is also real in the S plane. Eq. (28) becomes

$$\frac{1}{1+\alpha_d} < k_i(s) < \frac{1}{1-\alpha_d} \tag{30}$$

Thus the gain margin is given from Eq. (30) as

$$\mathrm{GM} = \frac{1}{1 \pm \alpha_d}, \qquad \text{for } \alpha_d \le 1 \tag{31}$$

Similarly, to get phase margin, let

$$l_i'(z) = e^{j\phi i'(z)} \tag{32}$$

then Eq. (23) becomes

$$|l_i'(z) - 1| < \alpha_d \tag{33}$$

Inverse Z transformation leads Eq. (33) to the following:

$$|l_i(s) * \frac{1-e^{-sT}}{s} - 1| < \alpha_d \tag{34}$$

$$|l_i(s) - 1| < \alpha_d \tag{35}$$

and Eq. (32) to

$$l_i(s) = e^{j\phi(s)} \tag{36}$$

Putting Eq. (36) into Eq. (35) yields

$$|e^{-j\phi(s)} - 1| < \alpha_d \tag{37}$$

equivalently,

$$|\phi_i'(s)| \cos^{-1}[1-(\alpha_d{}^2/2)] \tag{38}$$

Thus we get the phase margin as

$$\mathrm{PM} = \pm \cos^{-1}[1-(\alpha_d{}^2/2)] \tag{39}$$

The previous results of the relationships between the minimum SV of the system RDM and the gain and phase margins are identical to those of the continuous case.

Implementation of the SVG Calculation Method

Let the continuous representation and the equivalent representation of a MIMO plant be described as follows.

Continuous representation — Discrete representation

$$\begin{aligned} \dot{X}s(t) &= FXs(t) + GuU(t) \rightarrow & Xs(n+1) &= \Phi sXs(n) + \Gamma sU(n) \\ Z(t) &= HXs(t) & Z(n) &= HXs(n) \end{aligned}$$

where[13]

$$\Phi s = e^{FT} = \sum_{k=0}^{\infty} \frac{(FT)^k}{k!}, \qquad \Gamma s = \sum_{k=0}^{\infty} \frac{(FT)^k}{(k+1)!} T\,Gu$$

and the controller state equation is

$$Xc(n+1) = AcXc(n) + BcZ(n)$$

$$U(n) = CcXc(n) + DcZ(n) + R(n)$$

Combining plant and controller equations yields

$$X(n+1) = \Phi X(n) + \Gamma U(n) \tag{40a}$$

$$U(n) = -\xi X(n) + R(n) \tag{40b}$$

where

$$X = \begin{bmatrix} Xs \\ Xc \end{bmatrix}, \; \Phi = \begin{bmatrix} \Phi s & 0 \\ BcH & Ac \end{bmatrix}, \; \Gamma = \begin{bmatrix} \Gamma s \\ 0 \end{bmatrix}, \; \xi = [-DcH : -Cc]$$

The Z transformation of Eq. (40) mutatis mutandis gives

$$U(Z) = [I + \xi(IZ - \Phi)^{-1}\Gamma]^{-1} R(Z) \tag{41}$$

and RDM broken at plant input,

$$\begin{aligned} \mathrm{RDM} &= I + \xi(IZ-\Phi)^{-1}\Gamma \\ &= I + \xi\psi\Gamma \end{aligned} \tag{42}$$

where $\psi = (Iz - \Phi)^{-1}$.

Using continuous Eqs. (13) gives SVG equations with respect to an element in system matrices each as

$$\frac{\partial \sigma_i}{\partial p_\Phi} = \mathrm{Re} \cdot \mathrm{tr}\left[\xi\psi \frac{\partial \Phi}{\partial p_\Phi} \psi\Gamma V_i U_i^* \right] \tag{43a}$$

$$\frac{\partial \sigma_i}{\partial p_\Gamma} = \mathrm{Re} \cdot \mathrm{tr}\left[\xi\psi \frac{\partial \Gamma}{\partial p_\Gamma} V_i U_i^* \right] \tag{43b}$$

$$\frac{\partial \sigma_i}{\partial p_\xi} = \mathrm{Re} \cdot \mathrm{tr}\left[\frac{\partial \xi}{\partial p_\xi} \psi\Gamma V_i U_i^* \right] \tag{43c}$$

Now we are going to expand the preceding SVG equations to all elements in system matrices. Sensitivity information which we want to know is about continuous plant parameters which are in matrices F, Gu, and H and discrete controller parameters which are in matrices, Ac, Bc, and Cc. Thus Eqs. (43) are to be expressed with respect to these matrices.

The element of matrix F is in Φ and Γ only. Thus from Eqs. (43a) and (43b), we can give the expression for F as

$$\begin{aligned} \frac{\partial \sigma_i}{\partial F} &= \frac{\partial}{\partial F\Gamma}\{\mathrm{Re} \cdot \mathrm{tr}[\xi\psi\Gamma V_i U_i^*]\} + \frac{\partial}{\partial F\Phi}\{\mathrm{Re} \cdot \mathrm{tr}[\xi\psi\Phi\psi\Gamma V_i U_i^*]\} \\ &= \mathrm{Re} \cdot \left\{ \frac{\partial}{\partial F\Gamma} \mathrm{tr}[\xi\psi\Gamma V_i U_i^*] \right\} + \frac{\partial}{\partial F\Phi} \mathrm{tr}\{[\xi\psi\Phi\psi\Gamma V_i U_i^*]\} \end{aligned} \tag{44}$$

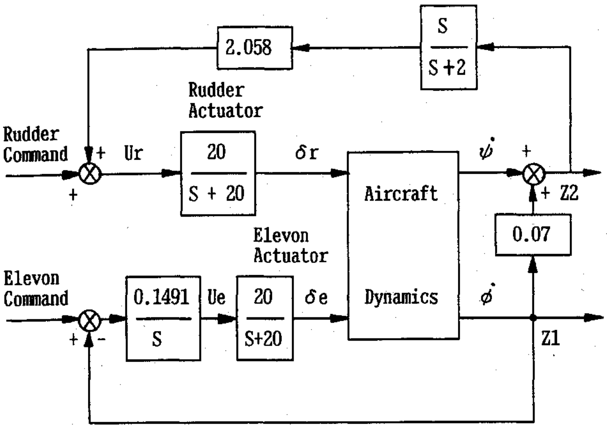

Fig. 3 Minimum singular value plot.

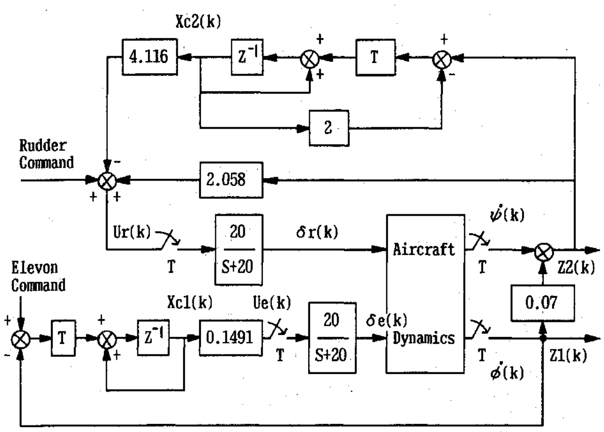

Fig. 4 SVG plot with respect to plant parameters in matrix F.

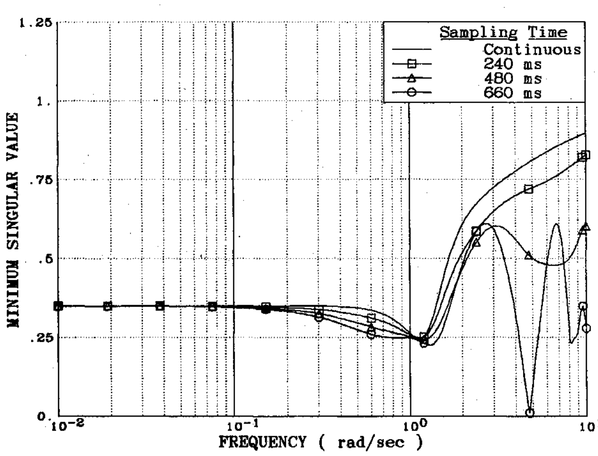

Fig. 5 SVG plot with respect to plant parameters in matrix Gu.

where $(\partial/\partial_{F_X})[\#]$ is the partial derivation with respect to the F, which is in X of #. The second part of the right-hand side of Eq. (44) is represented by

$$\frac{\partial}{\partial F\Phi}\operatorname{tr}[\xi\psi\Phi\psi\Gamma V_iU_i^*] = \frac{\partial}{\partial F\Phi}\operatorname{tr}\left[\xi\psi\begin{bmatrix}\Phi s\\ BcH\end{bmatrix}\psi\Gamma V_iU_i^*\right]$$

$$= \frac{\partial}{\partial F\Phi}\operatorname{tr}\left[\xi\psi\begin{bmatrix}e^{FT} & 0\\ 0 & 0\end{bmatrix}\psi\Gamma V_iU_i^*\right]$$

and using a matrix derivation equation (see the Appendix) of

$$\frac{\partial}{\partial(AT)}\operatorname{tr}[Be^{AT}C] = \left[\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{l=1}^{k}\{(AT)^{k-1}CB(AT)^{l-1}\}\right]^t \tag{45}$$

the second part is represented as

$$\frac{\partial}{\partial FT\Phi}\operatorname{tr}[\xi\psi\Phi\psi\Gamma V_iU_i^*]$$
$$= \left[\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{l=1}^{k}\{(FT)^{k-1}\bar{U}n^t\psi\Gamma V_iU_i^*\xi\psi(FT)^{l-1}\bar{U}n\}\right]^t \tag{46}$$

where

$$\bar{U}n = \begin{bmatrix}Un\\ 0_{mn}\end{bmatrix}$$

Un:$n \times n$ unit matrix, 0_{mn}:$m \times n$ null matrix, and $[\#]^t$: transpose of #. Similarly, the first term of the right-hand side of Eq. (44) is given by

$$\frac{\partial}{\partial FT\Gamma}\operatorname{tr}[\xi\psi\Gamma V_iU_i^*] = \left[\sum_{k=1}^{\infty}\frac{W}{(k+1)!}\sum_{l=1}^{k}(FT)^{k-1}\right.$$
$$\left.\times\, GuV_iU_i^*\xi\psi\bar{U}n(FT)^{l-1}\right]^t \tag{47}$$

Using Eqs. (44) and (45) finally gives the SVG equation with respect to F as

$$\frac{\partial\sigma_i}{\partial F} = T\frac{\partial\sigma_i}{\partial(FT)} = T\,\mathrm{Re}\cdot\left[\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{l=1}^{k}(FT)^{k-1}\right.$$
$$\left.\times\left(\bar{U}n^t\psi\Gamma + \frac{TGu}{k+1}\right)V_iU_i^*\xi\psi\bar{U}n(FT)^{l-1}\right]^t \tag{48}$$

Next, to obtain an SVG equation about elements of controller matrix Ac, we use Eq. (43a) (Ac exists in Φ only) and proceed as follows:

$$\frac{\partial\sigma_i}{\partial Ac} = \mathrm{Re}\cdot\left[\frac{\partial}{\partial Ac_\Phi}\operatorname{tr}\{\xi\psi\Phi\psi\Gamma V_iU_i^*\}\right]$$
$$= \mathrm{Re}\cdot\left[\frac{\partial}{\partial ACAC}\operatorname{tr}\left\{\xi\psi\begin{bmatrix}\Phi s & 0\\ BeH & Ac\end{bmatrix}\psi\Gamma V_iU_i^*\right\}\right]$$
$$= \mathrm{Re}\cdot\left[\frac{\partial}{\partial ACAC}\operatorname{tr}\left\{\xi\psi\begin{bmatrix}0\\ \bar{U}m\end{bmatrix}Ac[0{:}Um]\psi\Gamma V_iU_i^*\right\}\right]$$
$$= \mathrm{Re}\cdot[\bar{U}m^t\psi\Gamma V_iU_i^*\xi\psi\bar{U}m]^t \tag{49}$$

where

$$\bar{U}m = \begin{bmatrix}0nm\\ Um\end{bmatrix}$$

Um : $m \times m$ unit matrix, $0nm$: $n \times m$ null matrix.

Using a similar method gives Eqs. (50–53),

$$\frac{\partial\sigma_i}{\partial(BcH)} = \mathrm{Re}\cdot[\bar{U}n^t\psi\Gamma V_iU_i^*\xi\psi\bar{U}m]^t \tag{50}$$

$$\frac{\partial \sigma_i}{\partial Gu} = \mathrm{Re} \cdot \left[\sum_{k=0}^{\infty} \frac{T}{(k+1)!} V_i U_i^* \xi \psi Un(FT)^k \right]^t \quad (51)$$

$$\frac{\partial \sigma_i}{\partial(-DcH)} = \mathrm{Re} \cdot [\bar{U}n^t \psi \Gamma V_i U_i^*]^t \quad (52)$$

$$\frac{\partial \sigma_i}{\partial(-Cc)} = \mathrm{Re} \cdot [\bar{U}m^t \psi \Gamma V_i U_i^*]^t \quad (53)$$

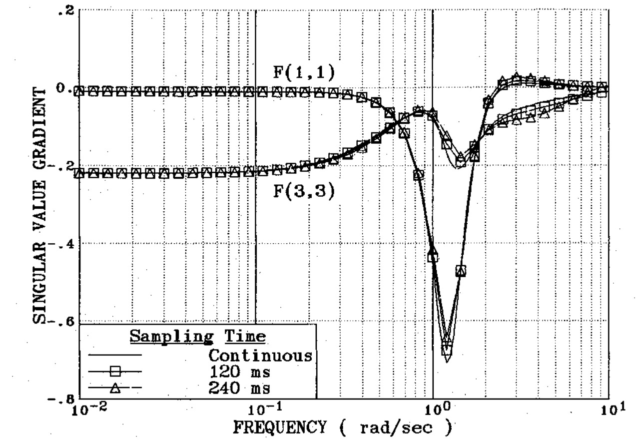

Fig. 6 SVG plot with respect to controller parameters in matrix C ($=Cc$).

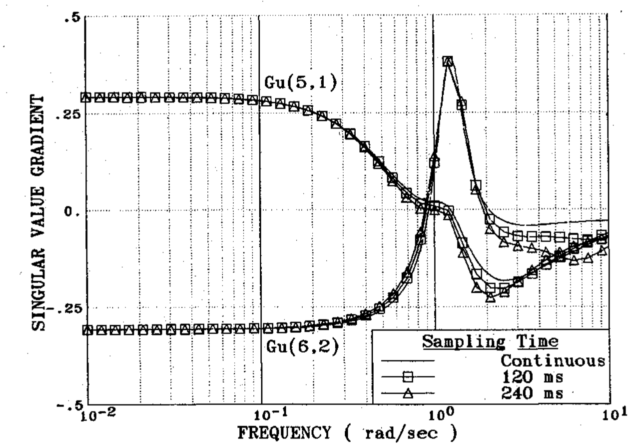

Fig. 7 Block diagram of the lateral attitude control system.

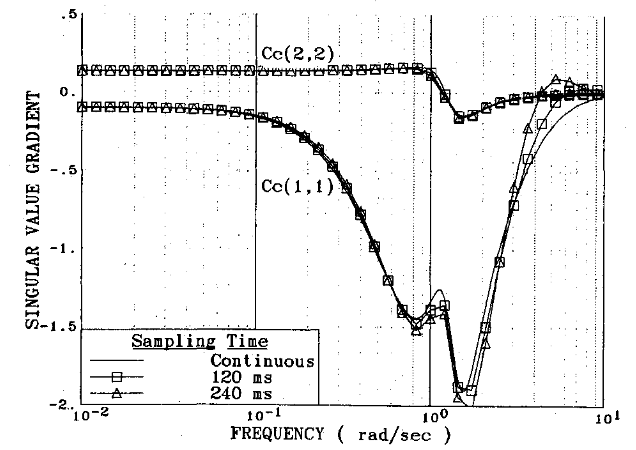

Fig. 8 Block diagram of the equivalent discrete system.

The derivatives with respect to H, Bc, and Dc are given in combined forms. Thus Eqs. (50) and (52) need to be decomposed as

$$\frac{\partial \sigma_i}{\partial H} = \mathrm{Re} \cdot [\bar{U}n^t \psi \Gamma V_i U_i^* (\xi \psi \bar{U}mBc - Dc)]^t \quad (54)$$

$$\frac{\partial \sigma_i}{\partial Bc} = \mathrm{Re} \cdot [H\bar{U}n^t \psi \Gamma V_i U_i^* \xi \psi \bar{U}m]^t \quad (55)$$

$$\frac{\partial \sigma_i}{\partial Dc} = \mathrm{Re} \cdot [H\bar{U}n^t \psi \Gamma V_i U_i^*]^t \quad (56)$$

Equations (48), (49), (51), and (53-56) derived previously are SVG calculation equations for a discrete system.

Example of the Sensitivity Analysis of Discrete Multiloop Systems

In this section, an application to a multiloop system is presented. The result is compared with the continuous case. The comparison is restricted to the plant parameters and the particular controller parameters which have the same values both in continuous and discrete forms. The procedure is applied to a remotely piloted vehicle lateral attitude control problem.[5] The block diagram is shown in Fig. 3. The system state equations are listed in Appendix B Table B1. The discrete version of the continuous case of the plant and the controller are represented in Fig. 4 and Appendix B Table B2. Using the SVG equations, which are listed in preceding sections, we get the sensitivity information for both continuous and discrete lateral loops in the interesting frequency range. In Fig. 5, the minimum SV of the RDM vs frequency is plotted. Figures 6 and 7 show the SVG values with respect to the plant parameter. Figure 8 shows the SVG values with respect to the controller parameter.

We can see the effects of sampling time in Figs. 3-6. The sampling time 240 ms is about 1/4 times the Nyquist sampling time of plant dynamics (the natural frequency of roll subsidence mode is 3.25 rad/s). Figure 3 shows that the system stability can be seriously affected by the slow sampling time and the stability of the example problem breaks down at the sampling time 660 ms. Although a conversion of the discrete controller to the continuous domain is possible, analyzing in continuous domain causes neglect of the sampling time effect; therefore one is not able to give the sensitivity information with respect to the discrete controller parameter. Thus the conversion method, which analyzes the discrete system in continuous domain after the conversion of the discrete system to the continuous domain, has some limitations. In Fig. 6 we can compare the results from both the continuous and discrete methods, because the Cc is equal to C. From Figs. 3-5, we can conclude that if the sampling interval is short enough, the analysis results from both the continuous and discrete domains are going to be close to each other. Thus the SVG equations derived previously in this paper are verified and are found to be useful.

Conclusions

A stability and sensitivity analysis method using singular value properties for discrete multiloop systems was described. Sensitivity analyses using a singular value gradient on a remotely piloted vehicle example were performed, and the results were coincident with the continuous case except for the effect of sampling time. From the comparison, a discrete version of the stability and sensitivity analysis method using singular value properties was verified. Thus, a singular-value based stability and sensitivity analysis tool for a discrete multiloop system is obtained.

Appendix A: Proof of Equation (45)

Using the matrix gradient equation,[11,12]

$$\frac{\partial}{\partial A}\operatorname{tr}[BAC] = [CB]^t \tag{A1}$$

we can proceed as

$$\frac{\partial}{\partial A}\operatorname{tr}[BA^2C] = \frac{\partial}{\partial A}\operatorname{tr}[BAAC]$$

$$= [ACB]^t + [CBA]^t$$

$$= [ACB + CBA]^t$$

$$\frac{\partial}{\partial A}\operatorname{tr}[BA^3C] = \frac{\partial}{\partial A}\operatorname{tr}[BAAAC]$$

$$= [A^2CB]^t + [ACBA]^t + [CBA^2]^t$$

$$= [A^2CB + ACBA + CBA^2]^t$$

and assume that the equation

$$\frac{\partial}{\partial A}\operatorname{tr}[BA^kC] = \left[\sum_{l=1}^{k} A^{k-1}CBA^{l-1}\right]^t \tag{A2}$$

is true. Then using Eqs. (A1) and (A2) gives Eq. (A3) as follows:

$$\frac{\partial}{\partial A}\operatorname{tr}[BA^{k+1}C] = \frac{\partial}{\partial A}\operatorname{tr}[BA^kAC]$$

$$= \left[\sum_{l=1}^{k} A^{k-1}ACBA^{l-1}\right]^t + [CBA^k]^t$$

$$= \left[\sum_{l=1}^{k} A^{k+1-1}CBA^{l-1}\right] + [A^{k+1-1}CBA^{l-1}|_{l=k+1}]^t$$

$$= \left[\sum_{l=1}^{k+1} A^{k+1-l}CBA^{l-1}\right]^t \tag{A3}$$

Adding $m = k + 1$, Eq. (A3) becomes

$$\frac{\partial}{\partial A}\operatorname{tr}[BA^mC] = \left[\sum_{l=1}^{m} A^{m-1}CBA^{l-1}\right]^t \tag{A4}$$

which is identical to Eq. (2). Thus Eq. (A2) is proved to be true.

Using the relationship of Eq. (A5),

$$\Phi = e^{FT} = \sum_{k=0}^{\infty} \frac{(FT)^k}{k!} \tag{A5}$$

and Eq. (A2), Eq. (45) can be proved such as

$$\frac{\partial}{\partial(AT)}\operatorname{tr}[Be^{AT}C] = \frac{\partial}{\partial(AT)}\operatorname{tr}\left[B\sum_{k=0}^{\infty}\frac{(AT)^k}{k!}C\right]$$

$$= \sum_{k=1}^{\infty}\frac{\partial}{\partial(AT)}\operatorname{tr}\left[B\frac{(AT)^k}{k!}C\right]$$

$$= \left\{\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{l=1}^{k}\left[(AT)^{k-1}CB\,(AT)^{l-1}\right]\right\}^t \tag{A6}$$

Appendix B: Lateral Attitude Control System of a Remotely Piloted Vehicle

Table B1 Plant and controller state equations and their matrices[5]

Plant: $Xs(t) = FXs(t) + GuU(t)$
$Z(t) = HXs(t)$

$Xs(t) = [\beta\ \phi\ \psi\ \phi\ \delta_e\ \delta_r]^t \qquad U(t) = [Ue\ Ur]^t$

$$F = \begin{vmatrix} -0.08527 & -0.0001423 & -0.9994 & 0.04142 & 0 & 0.1862 \\ -46.86 & -2.757 & -0.3896 & 0 & -124.3 & 128.6 \\ -0.4248 & -0.06224 & -0.06714 & 0 & -8.792 & -20.46 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.0 \end{vmatrix}$$

$$Gu = \begin{vmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad H = \begin{vmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0.07 & 1.0 & 0 & 0 & 0 \end{vmatrix}$$

Controller: $Xc(t) = AXc(t) + BZ(t)$
$U(t) = CXc(t) + DZ(t) + R(t)$

$$A = \begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix} \qquad B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} 0.1491 & 0 \\ 0 & -4.116 \end{vmatrix} \qquad D = \begin{vmatrix} 0 & 0 \\ 0 & 2.058 \end{vmatrix}$$

Appendix B (continued)

Table B2 Plant and controller state equations and their matrices that are converted from continuous form, for sampling time $T = 120$ ms

Plant: $Xs(k+1) = \Phi s Xs(k) + \Gamma s U(k)$
$Z(k) = HXs(k)$

$$\Phi_s = \begin{vmatrix} 0.9916 & 0.0007 & -0.1189 & 0.0049 & 0.0305 & 0.0862 \\ -4.7663 & 0.7170 & 0.3408 & 0.0125 & -4.5973 & 4.4710 \\ -0.0318 & -0.0063 & 0.9941 & 0.0001 & 0.3726 & -0.9527 \\ -0.3022 & 0.1021 & 0.0149 & 0.9995 & -0.4114 & 0.4151 \\ 0 & 0 & 0 & 0 & 0.0907 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0907 \end{vmatrix}$$

$$\Gamma_s = \begin{vmatrix} 0.0015 & 0.0043 \\ -0.4114 & 0.4151 \\ -0.0315 & -0.0773 \\ -0.0198 & 0.0202 \\ 0.0455 & 0.0 \\ 0.0 & 0.0455 \end{vmatrix} \qquad H = \begin{vmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0.07 & 1.0 & 0 & 0 & 0 \end{vmatrix}$$

Controller: $Xc(k+1) = AcXc(k) + BcZ(k)$
$Uc(k) = CcXc(k) + DcZ(n)$

$$Ac = \begin{vmatrix} 1 & 0 \\ 0 & 1-2T \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0.76 \end{vmatrix} \qquad Bc = \begin{vmatrix} T & 0 \\ 0 & T \end{vmatrix} = \begin{vmatrix} 0.12 & 0 \\ 0 & 0.12 \end{vmatrix}$$

$Cc = C \qquad Dc = D$

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